Novel Concepts for Constraint Treatments and Approximations in Efficient Structural Synthesis

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The paper presents some new concepts for efficient treatment of a large number of constraints. The ideas are based on finding effective ways of reducing the number of active constraints without degrading their overall behavioral characteristics or altering the essential nature of the original design problem. This has led to a set of three distinct and new concepts which are discussed here. The paper first presents a generalized concept of "dial-in" form of constraint approximation, which can supply a hierarchy of conforming constraint approximations—forms for increasing conservativeness and/or accuracy. It contains two tuning parameters, which can be used to control the quality of the constraint approximations for most structural types (truss, frame, panel, reinforced, etc.), and finite elements (membrane, bending, shear, plate, shell, etc.) of interest. Second, a concept of systematic partitioning the mixed constraint set into several distinct subsets is presented. This allows control of the constraint behaviors in a logical manner. Third, the paper introduces a concept of effectively collapsing the constraints that occur in packets of high concentrations. A determinate truss, two indeterminate trusses, and a thin beam are used as examples.

Introduction

IN recent years, several approximation concepts (see, e.g., Refs. 1-4) have been used to handle constraint and its derivatives in an effort to achieve efficient structural synthesis based on mathematical programming (MP) formulations. These approximation concepts include:

1) Temporary reduction of the number of inequality constraints by deletion techniques (e.g., regionalization, truncation).

2) Reduction of the number of finite element analyses by construction of explicit form of approximations (linear,[4] reciprocal,[2] or hybrid[4]) for the inequality constraints retained.

3) Use of design sensitivity techniques that reduce the number of forward backward substitutions (FBS) to the minimum of either the number of linked design variables or the number of active constraints (e.g., Ref. 5).

With the coordinated use of techniques 1-3, the efficiency of the current programs has shown marked improvement over past performance. Still, the programs are not efficient enough to solve a large-scale system such as an automobile body, helicopter, aircraft, naval ship, or spaceship. The factors which often contribute toward making solutions of large systems intractable are:

a) *Inadequate constraint treatment/reduction.* The available process of handling the large number of inequality constraints through item 1 can pose a severe computational burden on the design procedure, should the active constraints occur in large numbers. Additional means should be found to reduce the number of the active constraints (both behavioral and side constraints) so that the design process can become at least nearly independent of the number of constraints involved.

b) *Close-ended constraint approximations.* The linear and inverse constraint approximations, item 2, often used in recent algorithms are not general enough to handle complex situations found in multioperating environments (such as constraints induced due to bending, buckling, vibration, etc.). A library of dial-in forms of constraint approximation is most desirable to supply a continuous chain of approximate forms, depending upon the accuracy or the conservativeness required for the approximations.

c) *Inefficient derivative processing/handling.* Although derivative schemes such as the ones mentioned in item 3 are quite efficient, they could still pose severe computational burdens if derivatives of all the active constraints are required. Hence, it is highly desirable to propose methods that could further reduce the cost of derivatives.

This paper is therefore devoted to discussing these three major issues and to propose approaches (concepts) to overcome them. Each of the new concepts is addressed in separate sections and salient points are described individually. Three numerical examples are used to show the effectiveness of concepts I and II, and one additional example (25-bar truss) is used to show the advantages of concept III.

A General Optimization Formulation

Problem Statement

The general structural optimization problem can be stated as follows: minimize $F(v)$ subject to

$$g_k(v) \geq 0, \qquad k=1,\ldots,\bar{q} \tag{1}$$

and $v_{min} \leq v \leq v_{max}$; side constraints, where $F(v)$ is the mass of the structure, $g_k(v)$ the behavior constraints, and $\bar{q}$ the number of active constraints. v_{min} and v_{max} are the minimum and maximum bounds on the design variables. Using Eq. (1), a typical form of stress constraint, for example, can be expressed as

$$g = 1 - (\sigma/\sigma_{max}) \geq 0 \tag{2}$$

where σ_{max} is the allowable stress. The behavioral constraints in general are implicit functions of the vector of design variables v and are obtained by solving a finite element problem. Optimization problems of this kind [Eqs. (1) and (2)] have mostly been solved either using MP techniques or using recursive design methods based on optimality criteria.[1-10] The inherent generality of MP formulation to

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handle the varieties of constraint types frequently suffers from the large number of constraints that need to be considered for large-scale systems, making this approach sometimes quite time-consuming. Ordinarily, it is not essential to consider the side constraints along with the active sets of the behavioral constraints. A more general and convenient procedure results by considering the side constraints as a part of the design variable linking process.

Design Variable Linking with Self-Scaling and Side Constraint Eliminations

When design variable linking is used, one, or at most a few, independent design variables v_j control the structural parameter associated with all finite elements in that linking group. In some cases, a design parameter d_j such as thickness or area for an element is directly related to the design variable v_i as

$$d_j = \sum_{i=1}^{N} f_{ij} v_i; \qquad j=1,M \tag{3}$$

in which T is a vector of structural parameter (with components $d_1, d_2, \ldots, d_M$). The side constraints are specified as $d_{\min} \leq d \leq d_{\max}$. The form of Eq. (3) often requires the side constraint ($d_{\min} \leq d$ or $d \leq d_{\max}$) to be treated along with the other sets of the constraints. A more general form can be obtained by relating d_j as[3]:

$$d_j(v) = d_{0j} + \sum_{i=1}^{N} v_i f_{ij}; \qquad j=1,M$$

or

$$T = T_0 + [F]v \tag{4}$$

In Eq. (4), if T_0 is chosen a vector of minimum gage or reference parameter values, the limits on the side constraints can be expressed as the limits on the design variables; that is $v_{min} \leq v \leq v_{\max}$. If T_0 takes the value for $d_{\min}$, then $v_{\min}$ can be zero. It is no longer necessary to include the side constraints along with the remaining behavior constraints. The simple limits on the design variables, in turn, can be insured by modifying the one-dimensional search procedure.

Hierarchy of Conforming Explicit Constraint Approximations (Concept I)

In order to reduce the number of finite element analyses needed to obtain an optimum design, explicit approximations for the active constraints, $g_k(v)$, are usually generated.[1-4] In Ref. 9, the author developed a generalized concept of obtaining a family of constraint approximations that could possibly be derived based on first-order approximations. This includes the forms suited for linked design space, forms that are more accurate for bending or coupled (membrane + bending) plate finite elements, and a family of approximate forms where a varying degree of conservativeness in the approximations is required besides accuracy. They are based on the concepts of finding a suitable transmuted function space in which the constraints are more or less linearly behaved. This may serve, therefore, as a companion to the "intermediate" design variable concept proposed previously in Refs. 2-4.

Diversity of Applications

The need for conservativeness and accuracy stems, in part, from the need to be able to reduce iterations to optimize structures of various types, e.g., an automobile body, a large space truss, or an aircraft fuselage. Some are modeled using linear size-stiffness elements (such as membrane, bar, and shear panels) and others using nonlinear-size stiffness elements (such as bending, shell, or curved elements). The associated design problems are further complicated by the fact that a number of alternate load cases (acting in bending, axial, and torsion) have to be enforced. A few discrete forms of approximation may not be sufficiently general to satisfy all analysis needs (e.g., static, vibration, buckling) for all types of structures of interest and accurate enough to perform well during intermediate design moves. Consequently, a "dial-in" library of approximate forms is essential to provide both flexibility in choice and generality in applications.

Generalized Inverse-Power Approximation (GPA)

The conventional forms of constraint approximation (see, e.g., Refs. 2 and 4) are exact only for those types of constraints which exhibit a linear or reciprocal dependence on the design parameters. Most constraints (other than stress and displacements) and structural members (other than a simple truss) as discussed in Ref. 9 behave differently; it would not be precise to treat them in either form. These forms (linear and reciprocal) are also not suitable (see the Appendix) when a general form of the linking, Eq. (4), is used. It is more convenient to express the constraints in terms of the structural groups instead of design variables. Stipulating a Taylor's series expansion of constraints in terms of ϕ for all design parameters, g can be approximated at v as:

$$g(v) = g(v_0) + \sum_{k=1}^{M} [\phi_k(v) - \phi_{k0}(v_0)] \frac{\partial g(v_0)}{\partial \phi_k} + \frac{1}{2} \sum_{k=1}^{M} \sum_{\ell=1}^{M} (\phi_k - \phi_{k0})(\phi_\ell - \phi_{\ell 0}) \frac{\partial^2 g(v_0)}{\partial \phi_k \partial \phi_\ell} \tag{5}$$

where ϕ_k represents a relationship between the design variables v and the design parameters d_k forming the structural group k. M is the number of structural groups, i.e.,

$$\phi_k = f(v, d_k) \tag{6}$$

here ϕ is a transmuted function. A simple expression of ϕ_k, for example, can be d_k^{-3} or d_k^{-2} for a determinate flexural system such as cantilever beam, where d_k represents the thickness of the kth element group in the plane of bending.

In Ref. 9, two basic forms of generalized approximation, one based on linear expansion in d_k, i.e., $\phi_k = d_k(v)$ (GLA), and the other based on inverse approximation in d_k, i.e., $\phi_k = 1/d_k$ (GIA) were proposed. In the setting of the nomenclature defined earlier, a more versatile form of constraint approximation can be defined by using an expansion in ϕ, such that

$$\phi[d_k(v)] = \frac{1}{p-1} d_k^{1-p} \qquad \text{if } p \neq 1$$
$$= -\ell n [d_k] \qquad \text{if } p = 1 \tag{7}$$

where p is a controlling parameter, not necessarily an integer. Using the first two terms of the Taylor series expansion [Eq. (5)], the constraints at v can be expressed for any p as

$$g(v) = g(v_0) + \frac{1}{p-1} \sum_{i=1}^{N} \frac{\partial g}{\partial v_i}(v_0) \sum_{k=1}^{M} [d_k^{1-p}(v) - d_k^{1-p}(v_0)] \Big/ \frac{\partial \phi_k}{\partial v_i}; \qquad \text{for } p \neq 1 \tag{8}$$

and

$$g(v) = g(v_0) - \sum_{i=1}^{N} \frac{\partial g}{\partial v_i}(v_0) \sum_{k=1}^{M} \ell n \left[\frac{d_k(v)}{d_k(v_0)} \right] \Big/ \frac{\partial \phi_k}{\partial v_i}; \quad \text{for } p=1 \tag{9}$$

where

$$\frac{\partial \phi_k(v_0)}{\partial v_i} = -d_k^{-p} f_{ik} \tag{10}$$

Equations (8-10) constitute a new generalized form of the constraint approximation which is unified for a concise representation. A number of previous alternate forms can be derived from this representation by substituting different values of p. For example, with $p=0$, it leads to a linear expansion form of approximation (GLA), and with $p=2$, an inverse-expansion form (GIA).[9] For other values of p, several new inverse-power approximation (GPA) forms can be derived. For example, the value of $p=3$ leads to a new form of inverse-power approximation as

$$g(v) = g(v_0) - \frac{1}{2}\sum_{i=1}^{N}\frac{\partial g}{\partial v_i}(v_0)\sum_{k=1}^{M}\left[\frac{1}{d_k^2(v)} - \frac{1}{d_k^2(v_0)}\right]\frac{d_k^3(v_0)}{f_{ik}} \tag{11}$$

This was referred to in Ref. 9 as quadratic inverse-power approximation (QPA). Using this form, it can be checked that

$$g_{\text{GLA}} - g_{\text{GIA}} = \sum_{i=1}^{N}\frac{\partial g}{\partial v_i}\sum_{k=1}^{M}\frac{[d_k(v) - d_k(v_0)]^2}{f_{ik} d_k(v)} \tag{12}$$

and

$$g_{\text{GIA}} - g_{\text{QPA}} = \sum_{i=1}^{N}\frac{\partial g}{\partial v_i}(v_0)\sum_{k=1}^{M}\frac{d_k(v_0)}{2d_k^2(v)}\frac{[d_k(v) - d_k(v_0)]^2}{f_{ik}} \tag{13}$$

Since the coefficient multiplying $\partial g/\partial v_i$ in Eqs. (12) and (13) are positive for physical design parameters such as thickness and area, g_{GIA} will be more conservative than g_{GLA} for all g's that provide

$$\frac{\partial g(v_0)}{\partial v_i} \geq 0 \tag{14}$$

It is assumed that the coefficient matrix F consists of all positive elements. If some of its elements are negative, Eq. (14) can be replaced by

$$C_i\frac{\partial g(v_0)}{\partial v_i} \geq 0$$

where C_i represents the second summation term of Eq. (12). For positive C_i's, which is often the case, this implies

$$g_{\text{GLA}} > g_{\text{GIA}} \qquad \text{for } \frac{\partial g}{\partial v_i}(v_0) > 0 \tag{15}$$

also

$$g_{\text{GIA}} > g_{\text{QPA}} \qquad \text{for } \frac{\partial g}{\partial v_i}(v_0) > 0 \tag{16}$$

For negative C_i's, the reversals of the inequality in Eqs. (15) and (16) will hold. The two particular cases [Eqs. (15) and (16)] clearly define an existence of some behavioral similarity. This can be stated as:

$$g(p=0) > g(p=1) > g(p=2)\ldots; \quad \text{when } \frac{\partial g}{\partial v_i}(v_0) \geq 0 \tag{17}$$

or

$$g(p=p_{\max}) > g(p=p_{\max-1})\ldots; \quad \text{when } \frac{\partial g}{\partial v_i}(v_0) \leq 0 \tag{18}$$

where $p_{\min}$ is the minimum and $p_{\max}$ is the maximum value of p used for the constraint approximations. Thus, in GPA forms the variable p provides a means to control the degree of conservativeness required for the approximation.

Salient Remarks on Constraint Approximations (GPA)

1) The generalized power form of constraint approximation is versatile, for it allows the designer to dial in any suitable form which best fits a given description of the structure and its environment. With p being a real variable, there exist numerous possibilities for rationally selecting an appropriate pair (of $p_{\min}$, $p_{\max}$ set). Reference 9 describes typically how such p's can be selected.

2) The constraint approximation is hierarchical in nature. An increasingly more conservative estimate can be obtained by increasing the difference between the values of the pair ($p_{\min}$,$p_{\max}$) without changing its characteristic form. The distinctive traits of this form (GPA) lie in the two tuning parameters $p_{\min}$ and $p_{\max}$, which essentially control the state and quality of the constraint approximations.

3) The proposed approximation is based on finding a suitable transmuted function space in which constraints behave more or less linearly. This is considerably less expensive than the alternate approach of improving the quality by considering additional (second order) terms in the Taylor series.

Efficient Processing of Active Constraints (Concept II)

Most of the MP techniques, treat the active constraints globally irrespective of their types.[2-4] Although this reduces the burden of keeping track of individual constraints, it requires that each active constraint be treated individually or be assigned a given CA form, which will be used for all the constraints in the active set. With individual treatments, the derivative costs grow linearly with the number of constraints. Furthermore, a single form of CA may not be accurate for constraints of every type. It is thus suggested here to partition the constraints according to their types and the load cases. This can be done by appropriately classifying the constraints into several disjunct sets:

$$Q_{1j}, Q_{2j}, Q_{3j}, \ldots, Q_{mj} \qquad \text{for } m=1,\bar{m}; j=1,\bar{J} \tag{19}$$

and

$$\{g(\eta_{q_{mj}})\} \in Q_{mj}$$

so that each Q_{mj} contains constraints of the same type, say η (pure stiffness u, strength σ, frequency λ, modal amplitude ψ, buckling, stability constraints, etc.). Subscript m represents a constraint type and j, the load case. $\bar{m}$ denotes the number of constraint types and $\bar{j}$ the number of load cases used in the design. The sorted constraint vector is formed as

$$g_i = [\{g(u_{q_{11}})\}, \{g(\sigma_{q_{21}})\}, \{g(\lambda_{q_{31}})\} \ldots \{g(\eta_{q_{mj}})\} \ldots]^T \tag{20}$$

In Eq. (20) the constraints are arranged in order of constraint types for load case 1, 2, etc. The q's in Eq. (20) represent the number of active constraints found in the corresponding sets Q_{mj}. The required approximation for each set (representing a particular constraint type) can easily be obtained by accessing the library of possible dial-in forms (see the section on GPA), which best describes the behavior of this set.

Reduction of Active Constraints and Task of Computing Their Derivatives (Concept III)

Folding of Active Constraints

Let us suppose now that there exists after constraint deletions, $\bar{q}_{1j}, \bar{q}_{2j}, \bar{q}_{3j}, \ldots, \bar{q}_{mj}$ entries for each constraint type (m) and load case (j). The bars represent the reduced sets [see Eqs. (19) and (20)] which have survived the constraint deletion process:

$$\bar{q}_{mj} \leq q_{mj} \qquad \text{for } m=1,2,\ldots,\bar{m}; j=1,2,\ldots,\bar{J} \tag{21}$$

The net number of active constraints would then be

$$\bar{q} = \sum_{j=1}^{\bar{J}} \sum_{m=1}^{\bar{m}} \bar{q}_{mj} \tag{22}$$

In the design of large complex systems, the sum $\bar{q}$ could add up to a large number. Most of these constraints are implicit, and calculations of their design derivatives can be difficult and computationally expensive. If design derivatives are required for each and all of these active sets, it can result in numerical inefficiency.

It is recognized here that it is not essential to keep the identities of all of the individual active constraints forming a subset (say, $\bar{Q}_{mj}$) throughout an iteration. The genesis of the required trait can be borrowed from the spirit of penalty formulations[8,12] in which some measure of constraint violation is often employed. Hajela and Sobieski[13] have tried a quadratic exterior form of cumulative constraint idea first in connection with a controlled growth method and later by Sobieski[14] alone in his multilevel design work. The proposed concept, however, does not retain the desired characteristics of its member constraints.

The proposition here is to replace the standard approach of working with all the active constraints in an MP methodology by a smaller number of equivalent constraint sets. Furthermore, with the use of such equivalent sets, a design should converge in more or less the same number of iterations (the convergence should not deteriorate considerably). The forms of the conventional penalty functions cannot be used effectively for this purpose, since most of the forms do not degenerate to their original constraint if a single constraint is present or give value close to its original when multiple constraints of equal magnitudes are present. In order to achieve an equally good convergence rate with the reduced constraint sets, it is considered necessary here to introduce some rational criterion that effectively weighs the individual constraint contributions and at the same time preserves their basic characteristics. Such a measure is proposed here as

$$\Omega_{mjk} = \left[\sum_{i=1}^{\bar{q}_{mj}} \langle g_{ik} \rangle^b \right]^{1/b} \cdot H(g_k^-, g_k^+); \qquad b > 1 \tag{23}$$

where further distinctions are made on the constraints lying in feasible or infeasible space. For violated constraints, the angle brackets are formed as:

$$\langle g_{ik} \rangle = -g_i \text{ if } g_k^+ \leq g_i \leq g_k^- \leq \epsilon^- \leq 0$$
$$= 0 \qquad \text{if outside the above range} \tag{24a}$$

For nonviolated constraints, the angle brackets are formed as:

$$\langle g_{ik} \rangle = g_i \text{ if } g_k^+ \geq g_i \geq g_k^- \geq \epsilon^+ \geq 0$$
$$= 0 \quad \text{if outside the above range} \tag{24b}$$

and $H(g_k^-, g_k^+)$ is a Heaviside unit function defined as

$$H(g_k^-, g_k^+) = +1 \text{ if } g_k^- \text{ and } g_k^+ \geq 0$$
$$= -1 \text{ if } g_k^- \text{ and } g_k^+ \leq 0$$

where g_k^- and g_k^+ represent some closely spaced limits on g_i constituting a kth packet of an infeasible or feasible space and b is a real number that the user controls. The boundary points ϵ^+ and ϵ^- are usually assigned small values ($\leq \pm 0.05$ or smaller). They are here called constraint thickness parameters, since they encroach both the infeasible and feasible spaces. The cumulative form [Eq. (23)] is introduced here to include two important features:

1) The form effectively weighs the individual contributions of its member constraints.

2) It preserves the behavior of Ω similar to its member constraints.

The characteristics of Ω_{mjk} depend upon b, and it is shown later how this can be used to control the quality of constraint approximation and to satisfy the penalty requirement, if it has to be used in a penalty-based SUMT procedure.

Equation (23) thus allows us to piecewise fold the domain of the active constraints into several small and disjoint packets. Each packet (say kth) constitutes an equivalent but a distinct measure of the associated constraints spanning a pair (g_k^-, g_k^+):

$$\Omega(g_k^-, g_k^+) \subset \{ \bar{Q}_{mj}; g_{\min} \leq g_i \leq \epsilon^- \text{ or } \epsilon^+ \leq g_i \leq g_{\max} \} \tag{25}$$

where $g_{\min}$ and $g_{\max}$ are the minimum and maximum constraint values for mth type and jth load case. Such packets with the use of Eqs. (23) and (24) can be repeated independently for each constraint type, $m = 1,2,\ldots,\bar{m}$, a number of times each, depending upon the number of k divisions or pairs (g_k^-, g_k^+) assigned to each type. Note from Eq. (25) that constraints lying within a narrow region $(\epsilon^- < [g_i(v_0)] < \epsilon^+)$ have not been included in Ω, and therefore all the constraints in this range are treated individually.

Distinctive Features

The concept of this discrete weighted averaging of constraints appears to be quite beneficial particularly during the first few initial design cycles (or in a situation when the design is started from minimum gage point) since there could be many constraints that might have been present as violated (thus active) and each of them need not be considered independently. From a computational efficiency standpoint, it is also quite attractive, since for complex systems the number of implicit constraints requiring gradient calculations will be substantially reduced. The issues remaining to be answered are:

1) How one could then obtain the derivative of such a cumulative constraint without knowing the derivatives of its member constraints.

2) How one would then approximate this cumulative constraint functional during a typical line search process of an optimization algorithm (MP).

If both 1 and 2 can be overcome without any significant change in the net behavior of the original problem, the advantages of such concepts over the standard approach, of working with all the active constraints, are clearly noteworthy. Regarding item 1, Ref. 12 describes a technique for efficiently obtaining the derivative of a penalty function. Since the cumulative constraint differs manly in its representation, the same technique could be applied. The expressions for the derivatives and the details of the calculations for this cumulative form are contained in Ref. 15. It is shown there that the task of calculating derivatives for the active constraints lying within a constraint packet (with g^- and g^+ as boundary points) can be reduced to a single adjoint calculation for Ω.[15] The process eliminates the need to know the derivatives of the individual constraints to compute the derivatives of an equivalent constraint. Item 2 is discussed next.

Accuracy of Constraint Approximation for Ω_m

To determine whether the cumulative constraint form, Eq. (23), would sustain the same degree of accuracy as that of its individual constraints, Ω_{mjk} needs to be compared with its approximated value $\tilde{\Omega}_{mjk}$. To be realistic, however, two or more constraints, g_i, should be considered in each Ω_{mjk}. A single constraint forming Ω is equivalent to seeking an approximation of the same constraint (independent of b) and, therefore, the Taylor series expansion for Ω_{mjk} in a single constraint is exact to the same degree of accuracy as the original constraint itself. Without loss of any generality, the subscripts m, j, and k, in Eq. (23) may be suppressed and a new function Ω^+ can be defined as:

$$\Omega = \Omega^+ \cdot H(g^-, g^+) \tag{26}$$

where

$$\Omega^+ = \left[\sum_{i=1}^{\bar{q}} \langle g_i(v) \rangle^b \right]^{1/b}; \qquad b > 1 \tag{27}$$

Equation (27) represents an exact expression for $\Omega^+(v)$ evaluated at v. The exact values of the constraint and its derivatives at the initial point v_0 can be expressed as:

$$\Omega^+(v_0) = \left[\sum_{i=1}^{\bar{q}} \langle g_i(v_0) \rangle^b \right]^{1/b}; \qquad b > 1 \tag{28}$$

and

$$\frac{\partial \Omega^+(v_0)}{\partial v_j} = \frac{1}{[\Omega^+(v_0)]^{b-1}} \sum_{i=1}^{\bar{q}} \langle g_i(v_0) \rangle^{b-1} \frac{\partial \langle g_i \rangle}{\partial v_j} \tag{29}$$

Having known the constraint value and its first derivatives at v_0, the constraint function Ω at v can be approximated using a first-order Taylor series. The localized behavior of Ω in constraint space when approximated may be different than those of the individual constraints forming this set. However, since the set includes only the constraints of the same kind, it is hoped that macroscopically Ω will not behave very differently than the g_i's which make this set. In order to evaluate how much difference there would be if Ω is assumed to follow the characteristics of g_i, the approximating form $\Omega^+(v)$ and approximated form $\tilde{\Omega}^+(v)$ need to be compared.

Let $\bar{p}$ represent a power for which an associated constraint g_i can be accurately estimated, then one can write using Eq. (8) and $F_{ij} = \delta_{ij}$

$$\tilde{g}_i(v) = g_i(v_0) - \frac{1}{\bar{p}-1} \left[\sum_{j=1}^{N} \frac{\partial g_i}{\partial v_j}(v_0) \, [v_j^{1-\bar{p}} - v_{0j}^{1-\bar{p}}] \, v_{0j}^{\bar{p}} \right] \tag{30}$$

In stress constraints, for example with membrane plate, shear, or rod elements, it has been shown[9] that $\bar{p} = 2$ gives close to an exact value, and with plate bending elements $\bar{p} = 3$ gives close to an exact value. For a given constraint and a structural type, it is possible, therefore, to find a value for $\bar{p}$ that satisfies the equality

$$g_i(v) = \tilde{g}_i(v) \tag{31}$$

in most cases. If the same value of $\bar{p}$ is deemed applicable for Ω^+, cumulative constraint functional $\Omega^+(v)$ can be approximated similarly as

$$\tilde{\Omega}^+(v) = \Omega^+(v_0) - \frac{1}{(\bar{p}-1)} \sum_{j=1}^{N} \frac{\partial \Omega^+}{\partial v_j}(v_0) \, [v_j^{1-\bar{p}} - v_{0j}^{1-\bar{p}}] \, v_{0j}^{\bar{p}} \tag{32}$$

By substituting the values of derivatives from Eq. (29) into Eq. (32), the result can be written as:

$$\tilde{\Omega}^+ = \Omega^+(v_0) - \frac{1}{(\bar{p}-1)[\Omega_0^+(v)]^{b-1}} \left[\sum_{j=1}^{N} \sum_{i=1}^{\bar{q}} \langle g_i(v_0) \rangle^{b-1} \times \frac{\partial \langle g_i \rangle}{\partial v_j} \cdot [v_j^{1-\bar{p}} - v_{0j}^{1-\bar{p}}] \, v_{0j}^{\bar{p}} \right] \tag{33}$$

$$= \frac{1}{[\Omega^+(v_0)]^{b-1}} \left[\sum_{i=1}^{\bar{q}} \langle g_i(v_0) \rangle^{b-1} \left\{ \langle g_i(v_0) \rangle - \frac{1}{(\bar{p}-1)} \sum_{j=1}^{N} \frac{\partial \langle g_i \rangle}{\partial v_j} \cdot [v_j^{1-\bar{p}} - v_{0j}^{1-\bar{p}}] \, v_{0j}^{\bar{p}} \right\} \right] \tag{34}$$

By substituting Eq. (30) into Eq. (34), Eq. (34) can be simplified as

$$\tilde{\Omega}^+(v) [\Omega^+(v_0)]^{b-1} = \sum_{i=1}^{\bar{q}} \langle \tilde{g}_i(v) \rangle \langle g_i(v_0) \rangle^{b-1} \tag{35}$$

Equation (35) represents an approximate and Eq. (27) represents an exact expression of the cumulative constraint at point v. For those constraint types for which an accurate estimate can easily be obtained, Eq. (35) can be written using Eq. (31) as

$$\tilde{\Omega}^+(v) [\Omega^+(v_0)]^{b-1} = \sum_{i=1}^{\bar{q}} \langle g_i(v) \rangle \langle g_i(v_0) \rangle^{b-1} \tag{36}$$

Equation (36) thus provides a criterion which can be used to determine the possible deviation the constraint $\tilde{\Omega}^+$ may have from its true values. To illustrate this further, let us suppose that there exists a $\hat{q}$ number of constraints forming a cumulative set [Eq. (25)], which happens to be all equal, that is,

$$|g_i| = |g_j| = \bar{g} \text{ for } \forall\ i \text{ and } j; \qquad i, j = 1, \hat{q} \tag{37}$$

It can be shown that for the aforementioned case

$$\tilde{\Omega}^+(v) \equiv \Omega^+(v) = (\hat{q})^{1/b} \bar{g}(v) \tag{38}$$

the Taylor approximation $\bar{\Omega}^+(v)$ appears to agree pointwise with the true values, $\Omega^+(v)$ of the constraints irrespective of the parameter b and the number of constraints forming the cumulative set. The aforementioned finding is very interesting, because it poses a great potential for reducing the cost by lumping the constraints wherever a close nesting of the constraint occurs.

Magnitude of Error Analysis

The individual constraints forming a packet [Eqs. (26) and (27)] may never be all the same, but could be partitioned in such a way that at the starting point v_0 they form a narrow band of high concentration:

$$|g_0^-| \le |g_i(v_0)| \le |g_0^+| \tag{39}$$

where g_0^- and g_0^+ are some closely spaced limits spanning a packet. The approximation for $\Omega^+(v)$, expressed in terms of known quantities at v_0, is given by Eq. (32). If this approximation is employed during a line search procedure for estimating $\Omega^+(v)$, a relationship between the bounds of the approximate cumulative set, $\tilde{\Omega}^+(v)$ and the member constraints, $g_i(v)$ can be found using Eq. (36). This is discussed in Ref. 15 in more detail.

Salient Remarks

The concept given in Eq. (23) does not require any special treatment and could be handled on the same basis as any of the other constraints in the set (as if it had resulted through a physical limit). In summary, the following remarks may seem in order:

1) For each cumulative packet, the design derivative vector and the constraint approximation need to be calculated only once per iteration.

2) The cumulative concept form, if applied in a densely populated narrow band, would not deteriorate the quality of the constraint approximation if the member constraints have been adequately estimated.

3) The task of obtaining a better approximation for Ω through this concept has been shifted to that for the member constraints, which is a problem of a relatively simpler kind and could easily be handled through rationale of constraint approximations described in the section covering the hierarchy of conforming explicit constraint approximations.

Numerical Experiments

To show the contributions of the individual concepts proposed here, two separate case studies were constructed.

In the first case study, attempts were made to show the results of the constraint approximations when concepts I and II were combined. Some typical experiments considered in this study were:

1) The variance of tuning parameter p in GPA forms and the influence of GPA forms in providing accuracy and conservativeness in constraint approximations. The study covers only stress and displacement constraints but includes structures with bending finite elements.

2) The effects of using different constraint approximation forms on the design procedure or, more specifically, the effects of p on the number of iterations. In this context, an iteration consists of performing a finite element analysis, evaluating the constraints and their derivatives at the initial design point, and finding a suitable unidimensional search based on GPA.

3) The association of different approximation forms of GPA for different constraint types and their effects. This may be important when constraints involved in a design process belong to multiple types (such as stress and deflection) and their behaviors with respect to the design parameters are not the same.

In the second case study, the form of constraint approximation was not changed but the parameters relating to the cumulative constraint functional (concept III) were varied. Some typical experiments used were 1) the effects of increasing the width of the boundary points ($|g_0^+ - g^-|$) forming a packet, and 2) the effect of changing the initial distribution patterns for the constraints on the convergence.

The PARS (programs for analysis and resizing of structures) system, which was developed originally by Haftka and Prasad[3] as a companion code to the SPAR finite element program is used in this study. The program is modified to include the new "dial-in" capability for constraint approximation in both the generalized power (GPA) and hybrid (GHA) forms (concept I). The program now also includes features of constraint elimination as a part of the linking process and constraint partitioning according to the types (concept II). The program already had the capability of generating simple linear (SLA) and inverse (SIA) forms of approximations. These features are used here in the present paper to compute results for SA forms and then to compare them with GA forms. Other features of the PARS program, such as constraint formations, inactive constraint deletion, analytical sensitivity computation, etc., have been used as they were originally present. The optimization procedure now includes a penalty based approach called Variable Penalty Method[8] which combines the SUMT with Newton's method. The constraint derivatives were computed analytically and thus were exact to the accuracy of finite element analysis.

Results of First Case Study

Three numerical example problems; a two-bar truss, a ten-bar truss, and a cantilever thin beam design, were used in this study. To eliminate any bias due to starting differences, all the parameters of a problem, such as the initial design point (v_0), the initial penalty constant (r), the cut-off constant (g_0) and cut-off constraint value ($g_{\lim}$) once selected for a run were kept constant and only the forms of GA or SA were varied. The results of the constraint approximations during the one-dimensional search when design variables were perturbed as

$$v_i = v_{i0} + \alpha s_i \tag{40}$$

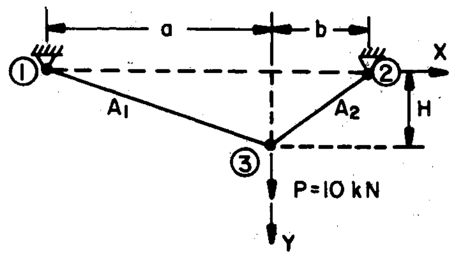

Fig. 1 Two-bar truss.

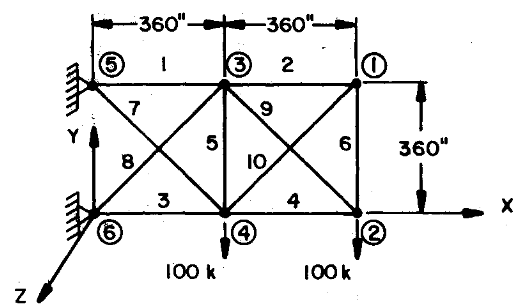

Fig. 2 Ten-bar truss.

Table 1 Comparison of constraint approximation results for two-bar truss example (during 1-D search)

1-D step No.	Critical constraint magnitudes stress (member = 1): Analysis FEA (exact)	Approximation GPA ($p=2$)	Approximation SIA Eq. (A3)	Starting parameters [Eqs. (4) and (A1)]
1	0.10557	0.10557	0.10557	
2	0.09649	0.09649	0.09651	$n=1$; $m=2$
3	0.08722	0.08722	0.08732	$v_i=1$; $i=1\rightarrow N$
4	0.07775	0.07775	0.07800	$d_{j0}=\{4,8\}^T$
5	0.06809	0.06809	0.06854	
6	0.05823	0.05823	0.05893	$f_{ij}=[1\ \ 2]$
7	0.04815	0.04815	0.04913	
8	0.03786	0.03786	0.03929	$d_{i0}^*=6$; $d_{\min}=4$
9	0.02734	0.02734	0.02924	1-D perturbation
				$\alpha=0.05027$
10	0.01658	0.01658	0.01904	
11	0.00559	0.00559	0.00868	

Table 2 Results of constraint approximations in 10-bar truss example (during 1-D search)

Run case No.	1-D step No.	Critical constraint magnitudes FEA (exact)	GPA ($p=2$)	SIA [Eq. (A3)]	Run case descriptions [Eq. (4) and Eq. (A1)]
	0	−0.2751	−0.2751	−0.2751	$v_i=10;\ i=1\rightarrow N$
	1	−0.2424	−0.2424	−0.2422	$d(v_0)=\{7,0.1,7,3,0.1,$
	2	−0.2114	−0.2114	−0.2107	$5,5,5,0.1\}^T$
1	3	−0.1820	−0.1820	−0.1806	$f_{ij}=\delta_{ij}$ and
	4	−0.1542	−0.1542	−0.1518	$d^*_{i0}=10;\ i=1\rightarrow N$
	5	−0.1278	−0.1278	−0.1242	1-D perturbation
	6	−0.1026	−0.1026	−0.0977	$\alpha=1.0157$
	7	−0.0786	−0.0786	−0.0723	
	0	0.0393	0.0393	0.0393	$v_i=15;\ i=1\rightarrow N$
	1	0.0394	0.0372	0.0379	$d(v_0)=\{7,0.1,7,3,0.1,0.1,$
2	2	0.0401	0.0304	0.0337	$5,5,5,0.1\}^T$
	3	0.0429	0.0236	0.0264	$f_{ij}=\delta_{ij}$ and
	4	0.0446	0.0328	—	$d^*_{i0}=10;\ i=1\rightarrow N$
					$\alpha=5.9981$
	0	0.02001	0.02001	0.02001	$v_i=20;\ i=1\rightarrow N$
	1	0.02015	0.01875	0.01875	$d_j(v_0)=0.1;\ j=1\rightarrow N$
3	2	0.02118	0.01477	0.01477	$f_{ij}=\delta_{ij}$ and
	3	0.02679	0.01486	0.01486	$d^*_{i0}=0.1;\ i=1\rightarrow N$
					$\alpha=7.6272$

are evaluated for each one-dimensional (1-D) step, α. The intermediate design steps are stored in the data base and later used to compute the corresponding exact values based on the finite element analysis (FEA). For brevity, only the critical constraint values at the starting design are given here in each constraint case. The equations, which are relevant to the results of the examples reported herein, are summarized in the Appendix.

Two-Bar Truss

The first example is an unsymmetrical two-bar truss[16] (Fig. 1) with one design variable controlling the areas of the two members in a ratio of 1:2. The linking parameters used in Eq. (4) were $N=1$, $M=2$, $v_0=1$, $d_j(v_0)=\{4,8\}$, $F=[1,2]$ for GA. The linking value for SA in Eq. (A1) was $d_i^*(v_0)=6$. The truss was designed for stress (allowables $=\pm10^4$ N/M^2) and minimum gage ($d_{\min}=4\text{m}^2$) constraints. The critical stress was found in member 1 ($g_{\min}=0.10557$). The results of constraint approximation found during the first 1-D search iteration are shown in Table 1. It may be noted that GPA ($p=2$) approximation reproduced the finite element analysis (FEA) results up to five significant decimal places.

Ten-Bar Truss

This is perhaps the most widely referred example in the literature. The nodes, elements, boundary conditions, and the loads are shown in Fig. 2. Three run cases have been considered for this example with $N=10$ and $M=10$. The linking parameter values used for each run case are given in Table 2. The truss was designed for stress (allowables $=\pm25$K) deflection (allowables $=2.0$, nodes $1\rightarrow4$, direction 2) and minimum gage $d_{\min}=0.1$) constraints. The deflection was the most critical constraint in each run case and occurred at node 2, direction 2. The results of the constraint approximation found during the first 1-D search iteration are shown in Table 2. It may be noted that GPA approximation compares very well with FEA results in each run case. In Run Case 3, as expected (see the Appendix) the SIA results agree with GPA ($p=2$) results.

Cantilever Thin Beam

The third example (Fig. 3) is selected from Ref. 17, for which analytical design is also known. The finite element

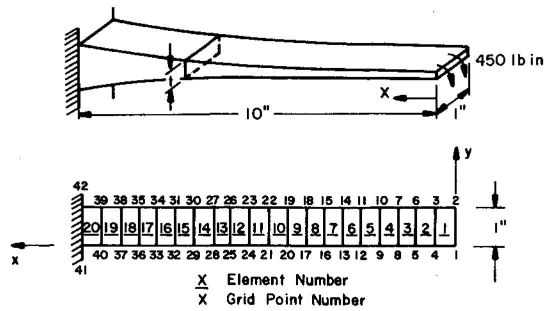

Fig. 3 Cantilever thin beam.

model was kept the same as given in Ref. 17. The parameters used in Eq. (4) were $N=20$, $M=20$, $v_{i0}=1.0$, $i=1\rightarrow N$, $d_j(v_0)=0.2, j=1\rightarrow M, f_{ij}=\delta_{ij}$ and $d_i^*(v_0)=1.0$; $i=1\rightarrow N$. The thin plate was designed for stress (allowables $=\pm30$K), deflection (allowables $=0.5$, nodes 1 and 3, direction 3) and minimum gage ($d_{\min}=0.2$) constraints. The most critical deflection constraint ($g_{\min}=0.97156$) occurred at node 1, direction 3. The most critical stress constraint ($g_{\min}=0.93749$) occurred in group 20 near the fixed end. The traces of the critical deflection and the critical stress constraints found during the 1-D search are shown in Tables 3a and 3b, respectively. The results of GPA (with $p=2$, 3, and 4), SIA, SLA, and FEA (exact) are also shown in Table 3 side by side for comparison. It is interesting to note that GPA ($p=4$) predicts exact values in the case of deflection constraint and GPA ($p=3$) predicts an exact matchable set in the case of stress. As shown, the higher the value of p used in GPA, the more conservative estimates it seemed to have predicted in each case.

Salient Remarks

1) The number of iterations depends on the form of constraint approximations. From Table 3, it may be noted that 50 to 60% improvements in the number of iterations can be achieved by using a correct (more accurate but slightly conservative) form of constraint approximation. For problems, where a more exact form of constraint approximations cannot be stipulated, the results may vary.

Table 3 Comparison of critical constraint results for cantilever thin beam example ($\alpha = 0.228696$)

	1-D step No.	FEA (exact)	Magnitudes of the critical constraint GPA power forms [Eq. (8)] $p=2$	$p=3$	$p=4$	Simple forms Eq. (A3) SIA	Eq. (A2) SLA
Deflection	0	0.9716	0.9716	0.9716	0.9716	0.9716	0.9716
	1	0.9676	0.9678	0.9677	0.9676	0.9678	0.9679
	2	0.9629	0.9637	0.9633	0.9629	0.9637	0.9643
	3	0.9573	0.9591	0.9582	0.9573	0.9591	0.9607
	4	0.9504	0.9542	0.9524	0.9504	0.9542	0.9571
	5	0.9419	0.9486	0.9456	0.9419	0.9486	0.9535
(A)	6	0.9315	0.9425	0.9376	0.9315	0.9425	0.9499
	7	0.9184	0.9356	0.9281	0.9184	0.9356	0.9463
	8	0.9017	0.9279	0.9167	0.9017	0.9279	0.9427
	9	0.8801	0.9190	0.9029	0.8801	0.9190	0.9390
	10	0.8517	0.9089	0.8859	0.8516	0.9089	0.9354
	11	0.8137	0.8972	0.8648	0.8135	0.8972	0.9318
	12	0.7612	0.8834	0.8379	0.7610	0.8834	0.9282
Stress	0	0.9375	0.9375	0.9375	0.9375	0.9375	0.9375
	1	0.9317	0.9321	0.9317	0.9286	0.9321	0.9341
	2	0.9251	0.9286	0.9251	0.9179	0.9286	0.9298
	3	0.9174	0.9230	0.9174	0.9050	0.9230	0.9270
	4	0.9085	0.9157	0.9085	0.8893	0.9157	0.9218
	5	0.8981	0.9069	0.8981	0.8698	0.9069	0.9167
(B)	6	0.8857	0.8963	0.8857	0.8455	0.8963	0.9081
	7	0.8711	0.8837	0.8711	0.8148	0.8837	0.8967
	8	0.8533	0.8685	0.8534	0.7753	0.8685	0.8910
	9	0.8317	0.8501	0.8318	0.7238	0.8501	0.8861
	10	0.8049	0.8275	0.8050	0.6553	0.8275	0.8812
	11	0.7712	0.7993	0.7713	0.5621	0.7993	0.8751
	12	0.7278	0.7601	0.7280	0.4320	0.7601	0.8713
No. of iterations		—	16	14	8	16	18

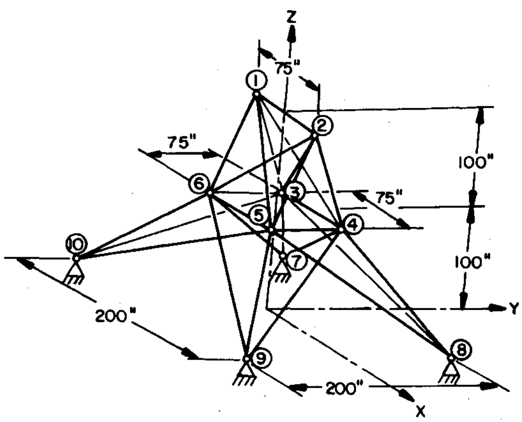

Fig. 4 Twenty-five bar truss.

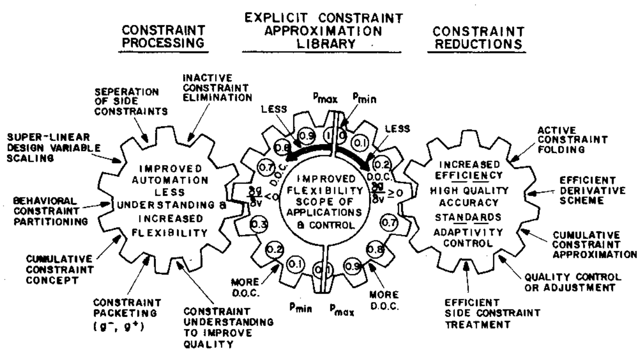

Fig. 5 Role of various concepts and components and their contributions in efficient structural synthesis.

2) The partition of the constraints according to their types (concept II) is important as one form of approximation may not be accurate for all. In Example 3, it is shown that for bending elements, a particular form of approximation is required for deflection constraint and another form for stress constraint to achieve maximum accuracy.

3) For truss finite elements, the GPA form with $p=2$ is more suitable than the SIA form. However, both forms may result in identical values if $m=n$ and $f_{ij}=\delta_{ij}$ (see the Appendix) are employed in Eq. (4).

Results of Second Case Study

In the second case study, a 25-member space truss problem (Fig. 4) is considered. The problem consists of eight independent design variables controlling the areas of 25 truss members (through linking) under two-load conditions and subject to stress, displacement, and minimum size limitations [0.0645 cm^2 ($\leq$0.01 in.^{2})]. The finite element model, loads, material properties, stress allowables, and the design variable linking descriptions were taken to be the same as originally used Ref. 6. The stress limits were specified in each member and the displacement limits of $\pm$8.89 mm (0.35 in.) were imposed on nodes 1 through 6 in the x, y, and z directions. Thus, there were a total of 25 stress constraints and 18 displacement constraints (without constraint deletions) for each load case.

The minimum weight designs were obtained using PARS. The constraint reductions are first accomplished using constraint deletion techniques followed by a cumulative constraint concept of the section covering folding active constraints (concept III). The constraint deletion is used to reduce the constraints into a smaller domain of active constraints ($g_i \leq g_{\lim}$) where $g_{\lim}$ represents a cutoff point in the constraint deletion process. In the implementation of the cumulative concept into PARS, during the packeting phase, the two boundary limits (g^-, g^+) of the packets were not independently assigned. The constraints, however, were collapsed by uniformly dividing the domain of the active constraints into packets of equal sizes, $\pm\Delta g$, and then collapsing the constraints into an equivalent constraint, where $\Delta g = |g^+ - g^-|$. A negative size value $(-\Delta g)$ was used for violated active constraints and $+\Delta g$ was used for nonviolated active constraints. The constraints lying in a small portion of the feasible and infeasible space $(\epsilon^- < g_i(v_0) < \epsilon^+)$ on either side of the zero constraint boundary $(g_i(v_0) = 0)$ were not collapsed, but treated individually. In summary, the constraints for feasible space were collapsed according to the packets spanned such as

$$(\epsilon^+, \epsilon^+ + \Delta g); (\epsilon^+ + \Delta g, \epsilon^+ + 2\Delta g); \ldots$$

and the constraints for infeasible space were collapsed according to the packets

$$(\epsilon^-, \epsilon^- - \Delta g); (\epsilon^- - \Delta g, \epsilon^- - 2\Delta g); \ldots$$

These reductions were carried out systematically for each constraint type (stress and displacements) and repeated for each load case. To nullify the constraint approximation effects on the results, a GPA form, once selected for this problem ($p = 2$), was not changed. To allow for constraint slack, two starting designs of the 25-bar truss problem are considered. In starting design 1, all members were assigned a cross-sectional area equal to 6.517 cm^2 (0.01 in.^{2}). In starting design 2, the iteration started from a point close to minimum gage; the cross-sectional areas of all members were equal to 0.129 cm^2 (0.02 in.^{2}).

To study the influence of Δg on the efficiency of the procedure, three run cases were formed for each starting design by choosing three distinctly separated values of Δg. In starting design 1, the most violated constraint observed was -1.19857 and the three selected values for Δg were (± 0.02, ± 0.2, and ± 1.00). In starting design 2, the most critical constraint found was -110.03 and the values selected for Δg were (± 0.25, ± 5.00, and ± 25.0). Tables 4 and 5 give the iteration and constraint counts for starting designs 1 and 2, respectively. The four entries of the constraint counts are for displacements and stresses, the first two (S_1, D_1) are obtained from load case 1 and the last two (S_2, D_2) from collapsing displacement and stress constraints in load case 2. As seen from this table, the penalty (measured in terms of increased number of iterations) is very small (one or two more) compared to the expected saving in the computational cost of the derivatives. The latter results from using the small number of so-called equivalent constraints shown in Tables 4 and 5.

It may be observed from Tables 4 and 5 that the sizes of the increments Δg have negligible effects. This is true, however, for the Δg values considered in the examples. The large value

Table 4 Iteration histories for 25-bar truss problem with concept III (starting design 1, $b = 10$, $\epsilon^+ = 0.02$, $\epsilon^- = -0.02$)

	History with equivalent constraints of different constraints increments, Δg									History with no equivalent constraints, original problem		
	$\Delta g = \pm 0.02$			$\Delta g = \pm 0.20$			$\Delta g = \pm 1.00$					
No. of iteration	Eq. critical constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb	Eq. critical constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb	Eq. critical constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb	Active constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb
0	11+19+7+10	47	334.03	5+8+4+7	24	334.03	2+3+2+3	10	334.03	18+25+18+25	86	334.03
1	10+14+6+12	42	622.14	4+4+3+5	16	617.85	3+2+1+2	8	615.33	18+25+18+25	86	651.11
2	9+14+6+11	40	644.39	4+4+3+5	16	621.94	3+2+1+2	8	620.28	18+25+18+25	86	606.43
3	9+15+7+11	42	622.04	4+5+3+5	17	604.50	3+2+1+2	8	602.38	18+25+18+25	86	592.07
4	8+12+8+10	38	609.69	4+4+3+4	15	610.84	3+2+1+2	8	608.62	4+0+2+0	6	584.56
5	14+16+7+10	47	588.50	4+4+3+4	15	584.15	3+2+1+2	8	578.24	4+0+6+0	10	561.20
6	2+0+2+0	4	567.13	2+0+2+0	4	574.66	2+0+2+0	4	566.15	2+0+2+0	4	547.13
7	2+0+2+0	4	558.24	2+0+2+0	4	563.93	2+0+2+0	4	560.85	2+0+2+0	4	547.99
8	2+0+2+0	4	551.23	2+0+2+0	4	556.87	2+0+2+0	4	554.43	2+0+2+0	4	546.71
9	2+0+2+0	4	546.03	2+0+2+0	4	550.65	2+0+2+0	4	548.45	2+0+2+0	4	545.85
10	2+0+2+0	4	545.18	2+0+2+0	4	545.50	2+0+2+0	4	545.45		376	
11		276		2+0+2+0	4	545.28	2+0+2+0	4	545.30			
					127			74				

S_1, S_2—stress constraints, local case 1 and 2. D_1, D_2—displacement constraints in local case 1 and 2.

Table 5 Iteration histories for 25-bar truss problem with concept III (starting design 2, $b = 10$, $\epsilon^+ = 0.02$, $\epsilon^- = -0.02$)

	History with equivalent constraints of different constraints increments, Δg									History with no equivalent constraints, original problem		
	$\Delta g = \pm 0.25$			$\Delta g = \pm 5.0$			$\Delta g = \pm 25.0$					
No. of iteration	Eq. critical constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb	Eq. critical constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb	Eq. critical constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb	Active constraints, $S_1+D_1+S_2+D_2$	Total No.	Weight, lb
0	15+24+9+13	61	6.61	6+10+7+8	31	6.61	4+4+4+4	16	6.61	18+25+18+25	86	6.61
1	11+21+7+12	51	58.75	3+5+5+4	17	66.26	2+3+2+2	9	78.53	18+25+18+25	86	65.93
2	5+6+7+7	25	343.76	2+1+2+1	6	389.33	2+1+2+1	6	457.19	18+25+18+25	86	473.60
3	3+3+3+4	13	695.76	3+1+1+1	6	611.39	3+1+1+1	6	613.35	18+25+18+25	86	742.18
4	4+4+2+3	13	625.74	3+1+1+1	6	580.95	3+1+1+1	6	612.20	18+25+18+25	86	615.55
5	4+4+4+4	16	563.49	3+1+3+1	8	557.85	3+1+1+1	6	581.75	18+25+18+25	86	600.24
6	4+2+4+2	12	564.92	3+1+3+1	8	551.70	3+1+3+1	8	568.53	18+25+18+25	86	576.90
7	4+2+4+2	12	551.21	3+1+3+1	8	548.27	3+1+3+1	8	557.93	18+25+18+25	86	556.42
8	2+0+2+0	4	550.23	2+0+2+0	4	546.57	2+0+2+0	4	545.87	2+0+2+0	4	551.26
9	2+0+2+0	4	546.81	2+0+2+0	4	545.29	2+0+2+0	4	545.45	2+0+2+0	4	546.63
10	2+0+2+0	4	545.54	2+0+2+0	4	545.08	2+0+2+0	4	545.24	2+0+2+0	4	545.27
11	2+0+2+0	4	545.12		102		2+0+2+0	4	545.29	2+0+2+0	4	545.09
		219						81			704	

of Δg as expected does affect the convergence trend. As a case of interest, in the case of starting design 2, a value of $\Delta g = \pm 100$ was employed (the maximum violated constraint was -110.03) for the 25-bar truss problem. At the end of 15 iterations (the set limit) the design failed to converge. This shows the necessity of assigning close bounds on Δg, when design is highly infeasible. For slightly infeasible designs, such restrictions may be relaxed in some problems. The latter remark is based on the result in Table 5 which is the case of starting design 1, in which a value of Δg as large as ± 1 (maximum violated constraint $= -1.19857$) was used and the results (measured as number of iterations) were not significantly different (11 versus 10).

Discussion and Conclusions

Some very useful concepts for achieving efficiency in the synthesis of large-scale structural systems are presented in this paper (see Fig. 5). They include: I) hierarchy of conforming explicit constraint approximations; II) efficient processing of the active constraints; and III) reduction of active constraints (both behavioral and side constraints) and the task of computing derivatives efficiently.

The parametric form of explicit constraint approximation is an important concept because, using this form, it is possible to unravel a higher degree of precision in the neighborhood estimates of the constraints. The domain of problem areas and reign of applicability can thus be extended to a number of new constraints, types of structures, and finite elements of interest. With some of the other new concepts introduced it is more attractive to use approximation concepts with constraints that may occur in a large number of active sets or that become large due to the presence of large numbers of conflicting constraint requirements (types). The cumulative constraint concept introduced in this paper provides a simple and logical means for reducing the number of active constraints. The characteristics that make this particularly appealing are the design sensitivity vectors and constraint approximations that need to be calculated for only the reduced number of equivalent constraints. The quality of such constraint (equivalent) approximation is also not difficult to control, as explained in the section covering the "reduction of active constraints and the task of computing their derivatives," if some rational criteria for constraint selections are initially met.

Some preliminary development and numerical examples are presented here. It is hoped that continued work and application experience in solving varieties of structural design problems would help define the requirements for extending the present techniques to an automated process.

Appendix: Difference Between GA and SA Forms

A general linking relationship, Eq. (4), was used to update the analysis in Ref. 3 in which M, N, and F_{ij} could take any value. During constraint approximation (linear SLA and inverse SIA), however, an approximate relationship was defined for structural parameters to be used during a one-dimensional search. A typical form used in Ref. 3 was:

$$d_i = d_{i0} + v_i; \qquad i = 1, N \tag{A1}$$

Eq. (A1) turns out to be a special case of Eq. (4) when $M = N$ and $F_{ij} = \delta_{ij}$. For cases when $M \neq N$ or when $F_{ij} \neq \delta_{ij}$, the values of d_{i0} in Eq. (A1) were adjusted to some nearest reference values. In SLA form, the constraints at the new design point v were approximated based on

$$g_L = g(v_0) + \sum_{i=1}^{N} [d_i(v) - d_i(v_0)] \frac{\partial g}{\partial v_i}(v_0) \tag{A2}$$

and in SIA form, the constraints were approximated using

$$g_I = g(v_0) - \sum_{i=1}^{N} \left[\frac{1}{d_i(v)} - \frac{1}{d_i(v_0)}\right] d_i^2(v_0) \frac{\partial g}{\partial v_i}(v_0) \tag{A3}$$

The forms of Eqs. (A2) and (A3) used in SA during 1-D search were, therefore, inconsistent with respect to the way the structural parameters were updated for the analysis. In GA forms, Eq. (4) is used at both places.

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